

# Credibility theory features of **actuar**

Christophe Dutang  
Université Paris Dauphine

Vincent Goulet  
Université Laval

Xavier Milhaud  
Université Claude Bernard Lyon 1

Mathieu Pigeon  
Université du Québec à Montréal

## 1 Introduction

Credibility models are actuarial tools to distribute premiums fairly among a heterogeneous group of policyholders (henceforth called *entities*). More generally, they can be seen as prediction methods applicable in any setting where repeated measures are made for subjects with different risk levels.

The credibility theory features of **actuar** consist of matrix `hachemeister` containing the famous data set of [Hachemeister \(1975\)](#) and function `cm` to fit hierarchical (including Bühlmann, Bühlmann-Straub), regression and linear Bayes credibility models. Furthermore, function `rcomphierarc` can simulate portfolios of data satisfying the assumptions of the aforementioned credibility models; see the "simulation" vignette for details.

## 2 Hachemeister data set

The data set of [Hachemeister \(1975\)](#) consists of private passenger bodily injury insurance average claim amounts, and the corresponding number of claims, for five U.S. states over 12 quarters between July 1970 and June 1973. The data set is included in the package in the form of a matrix with 5 rows and 25 columns. The first column contains a state index, columns 2–13 contain the claim averages and columns 14–25 contain the claim numbers:

```

> data(hachemeister)
> hachemeister
      state ratio.1 ratio.2 ratio.3 ratio.4 ratio.5
[1,]      1  1738   1642   1794   2051   2079
[2,]      2  1364   1408   1597   1444   1342
[3,]      3  1759   1685   1479   1763   1674
[4,]      4  1223   1146   1010   1257   1426
[5,]      5  1456   1499   1609   1741   1482
      ratio.6 ratio.7 ratio.8 ratio.9 ratio.10 ratio.11
[1,]   2234   2032   2035   2115   2262   2267
[2,]   1675   1470   1448   1464   1831   1612
[3,]   2103   1502   1622   1828   2155   2233
[4,]   1532   1953   1123   1343   1243   1762
[5,]   1572   1606   1735   1607   1573   1613
      ratio.12 weight.1 weight.2 weight.3 weight.4
[1,]   2517   7861   9251   8706   8575
[2,]   1471   1622   1742   1523   1515
[3,]   2059   1147   1357   1329   1204
[4,]   1306   407   396   348   341
[5,]   1690   2902   3172   3046   3068
      weight.5 weight.6 weight.7 weight.8 weight.9
[1,]   7917   8263   9456   8003   7365
[2,]   1622   1602   1964   1515   1527
[3,]   998   1077   1277   1218   896
[4,]   315   328   352   331   287
[5,]   2693   2910   3275   2697   2663
      weight.10 weight.11 weight.12
[1,]   7832   7849   9077
[2,]   1748   1654   1861
[3,]   1003   1108   1121
[4,]   384   321   342
[5,]   3017   3242   3425

```

### 3 Hierarchical credibility model

The linear model fitting function of R is `lm`. Since credibility models are very close in many respects to linear models, and since the credibility model fitting function of **actuar** borrows much of its interface from `lm`, we named the credibility function `cm`.

Function `cm` acts as a unified interface for all credibility models supported by the package. Currently, these are: the unidimensional models of [Bühlmann \(1969\)](#) and [Bühlmann and Straub \(1970\)](#); the hierarchical model of [Jewell \(1975\)](#) (of which the first two are special cases); the regression model of [Hachemeister \(1975\)](#), optionally with the intercept at the barycenter of time

(Bühlmann and Gisler, 2005, Section 8.4); linear Bayes models. The modular design of `cm` makes it easy to add new models if desired.

This section concentrates on usage of `cm` for hierarchical models.

There are some variations in the formulas of the hierarchical model in the literature. We compute the credibility premiums as given in Bühlmann and Jewell (1987) or Bühlmann and Gisler (2005), supporting three types of estimators of the between variance structure parameters: the unbiased estimators of Bühlmann and Gisler (2005) (the default), the slightly different version of Ohlsson (2005) and the iterative pseudo-estimators as found in Goovaerts and Hoogstad (1987) or Goulet (1998).

Consider an insurance portfolio where *entities* are classified into *cohorts*. In our terminology, this is a two-level hierarchical classification structure. The observations are claim amounts  $S_{ijt}$ , where index  $i = 1, \dots, I$  identifies the cohort, index  $j = 1, \dots, J_i$  identifies the entity within the cohort and index  $t = 1, \dots, n_{ij}$  identifies the period (usually a year). To each data point corresponds a weight — or volume —  $w_{ijt}$ . Then, the best linear prediction for the next period outcome of an entity based on ratios  $X_{ijt} = S_{ijt}/w_{ijt}$  is

$$\begin{aligned}\hat{\pi}_{ij} &= z_{ij}X_{ijw} + (1 - z_{ij})\hat{\pi}_i \\ \hat{\pi}_i &= z_iX_{izw} + (1 - z_i)m,\end{aligned}\tag{1}$$

with the credibility factors

$$\begin{aligned}z_{ij} &= \frac{w_{ij\Sigma}}{w_{ij\Sigma} + s^2/a}, & w_{ij\Sigma} &= \sum_{t=1}^{n_{ij}} w_{ijt} \\ z_i &= \frac{z_{i\Sigma}}{z_{i\Sigma} + a/b}, & z_{i\Sigma} &= \sum_{j=1}^{J_i} z_{ij}\end{aligned}$$

and the weighted averages

$$\begin{aligned}X_{ijw} &= \sum_{t=1}^{n_{ij}} \frac{w_{ijt}}{w_{ij\Sigma}} X_{ijt} \\ X_{izw} &= \sum_{j=1}^{J_i} \frac{z_{ij}}{z_{i\Sigma}} X_{ijw}.\end{aligned}$$

The estimator of  $s^2$  is

$$\hat{s}^2 = \frac{1}{\sum_{i=1}^I \sum_{j=1}^{J_i} (n_{ij} - 1)} \sum_{i=1}^I \sum_{j=1}^{J_i} \sum_{t=1}^{n_{ij}} w_{ijt} (X_{ijt} - X_{ijw})^2.\tag{2}$$

The three types of estimators for the variance components  $a$  and  $b$  are the

following. First, let

$$\begin{aligned} A_i &= \sum_{j=1}^{J_i} w_{ij\Sigma} (X_{ijw} - X_{iww})^2 - (J_i - 1)s^2 & c_i &= w_{i\Sigma\Sigma} - \sum_{j=1}^{J_i} \frac{w_{ij\Sigma}^2}{w_{i\Sigma\Sigma}} \\ B &= \sum_{i=1}^I z_{i\Sigma} (X_{izw} - \bar{X}_{zzw})^2 - (I - 1)a & d &= z_{\Sigma\Sigma} - \sum_{i=1}^I \frac{z_{i\Sigma}^2}{z_{\Sigma\Sigma}}, \end{aligned}$$

with

$$\bar{X}_{zzw} = \sum_{i=1}^I \frac{z_{i\Sigma}}{z_{\Sigma\Sigma}} X_{izw}. \quad (3)$$

(Hence,  $E[A_i] = c_i a$  and  $E[B] = d b$ .) Then, the Bühlmann–Gisler estimators are

$$\hat{a} = \frac{1}{I} \sum_{i=1}^I \max\left(\frac{A_i}{c_i}, 0\right) \quad (4)$$

$$\hat{b} = \max\left(\frac{B}{d}, 0\right), \quad (5)$$

the Ohlsson estimators are

$$\hat{a}' = \frac{\sum_{i=1}^I A_i}{\sum_{i=1}^I c_i} \quad (6)$$

$$\hat{b}' = \frac{B}{d} \quad (7)$$

and the iterative (pseudo-)estimators are

$$\tilde{a} = \frac{1}{\sum_{i=1}^I (J_i - 1)} \sum_{i=1}^I \sum_{j=1}^{J_i} z_{ij} (X_{ijw} - X_{izw})^2 \quad (8)$$

$$\tilde{b} = \frac{1}{I - 1} \sum_{i=1}^I z_i (X_{izw} - X_{zzw})^2, \quad (9)$$

where

$$X_{zzw} = \sum_{i=1}^I \frac{z_i}{z_{\Sigma}} X_{izw}. \quad (10)$$

Note the difference between the two weighted averages (3) and (10). See [Belhadj et al. \(2009\)](#) for further discussion on this topic.

Finally, the estimator of the collective mean  $m$  is  $\hat{m} = X_{zzw}$ .

The credibility modeling function `cm` assumes that data is available in the format most practical applications would use, namely a rectangular array (matrix or data frame) with entity observations in the rows and with one or more classification index columns (numeric or character). One will recognize the output format of `rcomphierarc` and its summary methods.

Then, function `cm` works much the same as `lm`. It takes in argument: a formula of the form `~ terms` describing the hierarchical interactions in a data set; the data set containing the variables referenced in the formula; the names of the columns where the ratios and the weights are to be found in the data set. The latter should contain at least two nodes in each level and more than one period of experience for at least one entity. Missing values are represented by NAs. There can be entities with no experience (complete lines of NAs).

In order to give an easily reproducible example, we group states 1 and 3 of the Hachemeister data set into one cohort and states 2, 4 and 5 into another. This shows that data does not have to be sorted by level. The fitted model below uses the iterative estimators of the variance components.

```
> X <- cbind(cohort = c(1, 2, 1, 2, 2), hachemeister)
> fit <- cm(~cohort + cohort:state, data = X,
+         ratios = ratio.1:ratio.12,
+         weights = weight.1:weight.12,
+         method = "iterative")
> fit
Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")

Structure Parameters Estimators

Collective premium: 1746

Between cohort variance: 88981
Within cohort/Between state variance: 10952
Within state variance: 139120026
```

The function returns a fitted model object of class `"cm"` containing the estimators of the structure parameters. To compute the credibility premiums, one calls a method of `predict` for this class.

```
> predict(fit)
$cohort
[1] 1949 1543

$state
[1] 2048 1524 1875 1497 1585
```

One can also obtain a nicely formatted view of the most important results with a call to `summary`.

```
> summary(fit)
Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")
```

#### Structure Parameters Estimators

Collective premium: 1746

Between cohort variance: 88981

Within cohort/Between state variance: 10952

Within state variance: 139120026

#### Detailed premiums

Level: cohort

	cohort	Indiv.	mean	Weight	Cred. factor	Cred. premium
1	1967		1.407	0.9196		1949
2	1528		1.596	0.9284		1543

Level: state

	cohort	state	Indiv.	mean	Weight	Cred. factor
1	1	2061		100155	0.8874	
2	2	1511		19895	0.6103	
1	3	1806		13735	0.5195	
2	4	1353		4152	0.2463	
2	5	1600		36110	0.7398	

Cred. premium

2048

1524

1875

1497

1585

The methods of predict and summary can both report for a subset of the levels by means of an argument levels.

```
> summary(fit, levels = "cohort")
```

Call:

```
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,  
    weights = weight.1:weight.12, method = "iterative")
```

#### Structure Parameters Estimators

Collective premium: 1746

Between cohort variance: 88981

Within cohort variance: 10952

#### Detailed premiums

```

      cohort Individ. mean Weight Cred. factor Cred. premium
      1      1967      1.407  0.9196      1949
      2      1528      1.596  0.9284      1543
> predict(fit, levels = "cohort")
$cohort
[1] 1949 1543

```

## 4 Bühlmann and Bühlmann–Straub models

As mentioned above, the Bühlmann and Bühlmann–Straub models are simply one-level hierarchical models. In this case, the Bühlmann–Gisler and Ohlsson estimators of the between variance parameters are both identical to the usual [Bühlmann and Straub \(1970\)](#) estimator

$$\hat{a} = \frac{w_{\Sigma\Sigma}}{w_{\Sigma\Sigma}^2 - \sum_{i=1}^I w_{i\Sigma}^2} \left( \sum_{i=1}^I w_{i\Sigma} (X_{iw} - X_{ww})^2 - (I-1)\hat{s}^2 \right), \quad (11)$$

and the iterative estimator

$$\tilde{a} = \frac{1}{I-1} \sum_{i=1}^I z_i (X_{iw} - X_{zw})^2 \quad (12)$$

is better known as the Bichsel–Straub estimator.

To fit the Bühlmann model using `cm`, one simply does not specify any weights.

```

> cm(~state, hachemeister, ratios = ratio.1:ratio.12)
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12)

Structure Parameters Estimators

Collective premium: 1671

Between state variance: 72310
Within state variance: 46040

```

When weights are specified together with a one-level model, `cm` automatically fits the Bühlmann–Straub model to the data. In the example below, we use the Bichsel–Straub estimator for the between variance.

```

> cm(~state, hachemeister, ratios = ratio.1:ratio.12,
+    weights = weight.1:weight.12)
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12)

```

#### Structure Parameters Estimators

Collective premium: 1684

Between state variance: 89639

Within state variance: 139120026

## 5 Regression model of Hachemeister

The credibility regression model of [Hachemeister \(1975\)](#) is a generalization of the Bühlmann–Straub model. If data shows a systematic trend, the latter model will typically under- or over-estimate the true premium of an entity. The idea of [Hachemeister](#) was to fit to the data a regression model where the parameters are a credibility weighted average of an entity's regression parameters and the group's parameters.

In order to use `cm` to fit a credibility regression model to a data set, one simply has to supply as additional arguments `regformula` and `regdata`. The first one is a formula of the form `~` terms describing the regression model, and the second is a data frame of regressors. That is, arguments `regformula` and `regdata` are in every respect equivalent to arguments `formula` and `data` of `lm`, with the minor difference that `regformula` does not need to have a left hand side (and is ignored if present). Below, we fit the model

$$X_{it} = \beta_0 + \beta_1 t + \varepsilon_t, \quad t = 1, \dots, 12$$

to the original data set of [Hachemeister \(1975\)](#).

```
> fit <- cm(~state, hachemeister, regformula = ~ time,
+         regdata = data.frame(time = 1:12),
+         ratios = ratio.1:ratio.12,
+         weights = weight.1:weight.12)
> fit
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, regformula = ~time, regdata = data.frame(time = 1:12))
```

#### Structure Parameters Estimators

Collective premium: 1469 32.05

Between state variance: 24154 2700.0  
2700 301.8

Within state variance: 49870187

To compute the credibility premiums, one has to provide the “future” values of the regressors as in `predict.lm`.



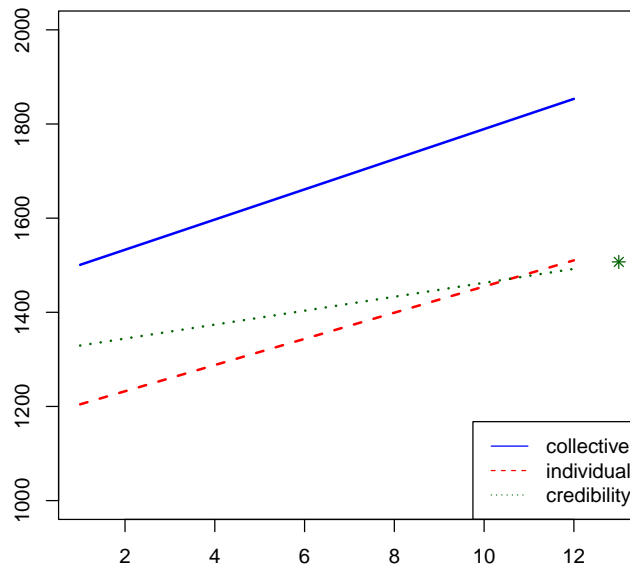


Figure 1: Collective, individual and credibility regression lines for State 4 of the Hachemeister data set. The point indicates the credibility premium.

```
> predict(fit, newdata = data.frame(time = 13))
[1] 2437 1651 2073 1507 1759
```

It is well known that the basic regression model has a major drawback: there is no guarantee that the credibility regression line will lie between the collective and individual ones. This may lead to grossly inadequate premiums, as Figure 1 shows.

The solution proposed by [Bühlmann and Gisler \(1997\)](#) is simply to position the intercept not at time origin, but instead at the barycenter of time (see also [Bühlmann and Gisler, 2005](#), Section 8.4). In mathematical terms, this essentially amounts to using an orthogonal design matrix. By setting the argument `adj.intercept` to `TRUE` in the call, `cm` will automatically fit the credibility regression model with the intercept at the barycenter of time. The resulting regression coefficients have little meaning, but the predictions are sensible.

```
> fit2 <- cm(~state, hachemeister, regformula = ~ time,
+           regdata = data.frame(time = 1:12),
+           adj.intercept = TRUE,
```

```

+         ratios = ratio.1:ratio.12,
+         weights = weight.1:weight.12)
> summary(fit2, newdata = data.frame(time = 13))
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
   weights = weight.1:weight.12, regformula = ~time, regdata = data.frame(time = 1:12),
   adj.intercept = TRUE)

Structure Parameters Estimators

Collective premium: -1675 117.1

Between state variance: 93783    0
                        0 8046
Within state variance: 49870187

Detailed premiums

state Indiv. coef. Cred. matrix Adj. coef.
1      -2062.46    0.9947 0.0000 -2060.41
      216.97    0.0000 0.9413  211.10
2     -1509.28    0.9740 0.0000 -1513.59
      59.60    0.0000 0.7630   73.23
3     -1813.41    0.9627 0.0000 -1808.25
      150.60    0.0000 0.6885  140.16
4     -1356.75    0.8865 0.0000 -1392.88
      96.70    0.0000 0.4080  108.77
5     -1598.79    0.9855 0.0000 -1599.89
      41.29    0.0000 0.8559   52.22
Cred. premium
2457

1651

2071

1597

1698

```

Figure 2 shows the beneficial effect of the intercept adjustment on the premium of State 4.

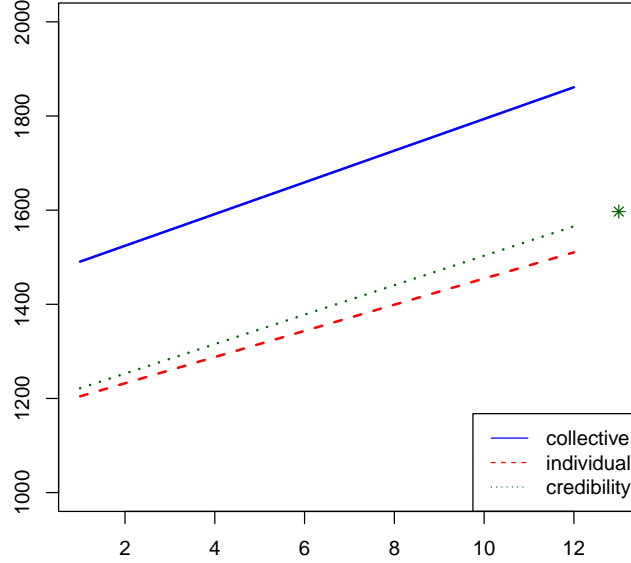


Figure 2: Collective, individual and credibility regression lines for State 4 of the Hachemeister data set when the intercept is positioned at the barycenter of time. The point indicates the credibility premium.

## 6 Linear Bayes model

In the pure bayesian approach to the ratemaking problem, we assume that the observations  $X_t$ ,  $t = 1, \dots, n$ , of an entity depend on its risk level  $\theta$ , and that this risk level is a realization of an unobservable random variable  $\Theta$ . The best (in the mean square sense) approximation to the unknown risk premium  $\mu(\theta) = E[X_t | \Theta = \theta]$  based on observations  $X_1, \dots, X_n$  is the Bayesian premium

$$B_{n+1} = E[\mu(\Theta) | X_1, \dots, X_n].$$

It is then well known (Bühlmann and Gisler, 2005; Klugman et al., 2012) that for some combinaisons of distributions, the Bayesian premium is linear and can written as a credibility premium

$$B_{n+1} = z\bar{X} + (1 - z)m,$$

where  $m = E[\mu(\Theta)]$  and  $z = n / (n + K)$  for some constant  $K$ .

The combinations of distributions yielding a linear Bayes premium involve

members of the univariate exponential family for the distribution of  $X|\Theta = \theta$  and their natural conjugate for the distribution of  $\Theta$ :

- $X|\Theta = \theta \sim \text{Poisson}(\theta), \Theta \sim \text{Gamma}(\alpha, \lambda)$ ;
- $X|\Theta = \theta \sim \text{Exponential}(\theta), \Theta \sim \text{Gamma}(\alpha, \lambda)$ ;
- $X|\Theta = \theta \sim \text{Normal}(\theta, \sigma_2^2), \Theta \sim \text{Normal}(\mu, \sigma_1^2)$ ;
- $X|\Theta = \theta \sim \text{Bernoulli}(\theta), \Theta \sim \text{Beta}(a, b)$ ;
- $X|\Theta = \theta \sim \text{Geometric}(\theta), \Theta \sim \text{Beta}(a, b)$ ;

and the convolutions

- $X|\Theta = \theta \sim \text{Gamma}(\tau, \theta), \Theta \sim \text{Gamma}(\alpha, \lambda)$ ;
- $X|\Theta = \theta \sim \text{Binomial}(v, \theta), \Theta \sim \text{Beta}(a, b)$ ;
- $X|\Theta = \theta \sim \text{Negative Binomial}(r, \theta)$  and  $\Theta \sim \text{Beta}(a, b)$ .

[Appendix A](#) provides the complete formulas for the above combinations of distributions.

In addition, [Bühlmann and Gisler \(2005, section 2.6\)](#) show that if  $X|\Theta = \theta \sim \text{Single Parameter Pareto}(\theta, x_0)$  and  $\Theta \sim \text{Gamma}(\alpha, \lambda)$ , then the Bayesian estimator of parameter  $\theta$  — not of the risk premium! — is

$$\hat{\Theta} = \eta \hat{\theta}^{\text{MLE}} + (1 - \eta) \frac{\alpha}{\lambda},$$

where

$$\hat{\theta}^{\text{MLE}} = \frac{n}{\sum_{i=1}^n \ln(X_i/x_0)}$$

is the maximum likelihood estimator of  $\theta$  and

$$\eta = \frac{\sum_{i=1}^n \ln(X_i/x_0)}{\lambda + \sum_{i=1}^n \ln(X_i/x_0)}$$

is a weight not restricted to  $(0, 1)$ . (See the "distributions" package vignette for details on the Single Parameter Pareto distribution.)

When argument formula is "bayes", function `cm` computes pure Bayesian premiums — or estimator in the Pareto/Gamma case — for the combinations of distributions above. We identify which by means of argument likelihood that must be one of "poisson", "exponential", "gamma", "normal", "bernoulli", "binomial", "geometric", "negative binomial" or "pareto". The parameters of the distribution of  $X|\Theta = \theta$ , if any, and those of the distribution of  $\Theta$  are specified using the argument names (and default values) of `dgamma`, `dnorm`, `dbeta`, `dbinom`, `dnbinom` or `dpareto1`, as appropriate.

Consider the case where

$$\begin{aligned} X|\Theta = \theta &\sim \text{Poisson}(\theta) \\ \Theta &\sim \text{Gamma}(\alpha, \lambda). \end{aligned}$$

The posterior distribution of  $\Theta$  is

$$\Theta|X_1, \dots, X_n \sim \text{Gamma} \left( \alpha + \sum_{t=1}^n X_t, \lambda + n \right).$$

Therefore, the Bayesian premium is

$$\begin{aligned} B_{n+1} &= E[\mu(\Theta)|X_1, \dots, X_n] \\ &= E[\Theta|X_1, \dots, X_n] \\ &= \frac{\alpha + \sum_{t=1}^n X_t}{\lambda + n} \\ &= \frac{n}{n + \lambda} \bar{X} + \frac{\lambda}{n + \lambda} \frac{\alpha}{\lambda} \\ &= z\bar{X} + (1 - z)m, \end{aligned}$$

with  $m = E[\mu(\Theta)] = E[\Theta] = \alpha/\lambda$  and

$$z = \frac{n}{n + K}, \quad K = \lambda.$$

One may easily check that if  $\alpha = \lambda = 3$  and  $X_1 = 5, X_2 = 3, X_3 = 0, X_4 = 1, X_5 = 1$ , then  $B_6 = 1.625$ . We obtain the same result using `cm`.

```
> x <- c(5, 3, 0, 1, 1)
> fit <- cm("bayes", x, likelihood = "poisson",
+          shape = 3, rate = 3)
> fit
Call:
cm(formula = "bayes", data = x, likelihood = "poisson", shape = 3,
   rate = 3)

Structure Parameters Estimators

Collective premium: 1

Between variance: 0.3333
Within variance: 1
> predict(fit)
[1] 1.625
> summary(fit)
Call:
cm(formula = "bayes", data = x, likelihood = "poisson", shape = 3,
   rate = 3)

Structure Parameters Estimators
```

Collective premium: 1

Between variance: 0.3333

Within variance: 1

Detailed premiums

Indiv.	mean	Weight	Cred. factor	Bayes premium
2		5	0.625	1.625

## A Linear Bayes formulas

This appendix provides the main linear Bayes credibility results for combinations of a likelihood function member of the univariate exponential family with its natural conjugate. For each combination, we provide, other than the names of the distributions of  $X|\Theta = \theta$  and  $\Theta$ :

- the posterior distribution  $\Theta|X_1 = x_1, \dots, X_n = x_n$ , always of the same type as the prior, only with updated parameters;
- the risk premium  $\mu(\theta) = E[X|\Theta = \theta]$ ;
- the collective premium  $m = E[\mu(\Theta)]$ ;
- the Bayesian premium  $B_{n+1} = E[\mu(\Theta)|X_1, \dots, X_n]$ , always equal to the collective premium evaluated at the parameters of the posterior distribution;
- the credibility factor when the Bayesian premium is expressed as a credibility premium.

### A.1 Bernoulli/beta case

$X|\Theta = \theta \sim \text{Bernoulli}(\theta)$

$\Theta \sim \text{Beta}(a, b)$

$\Theta|X_1 = x_1, \dots, X_n = x_n \sim \text{Beta}(\tilde{a}, \tilde{b})$

$$\tilde{a} = a + \sum_{t=1}^n x_t$$

$$\tilde{b} = b + n - \sum_{t=1}^n x_t$$

Risk premium

$$\mu(\theta) = \theta$$

Collective premium

$$m = \frac{a}{a+b}$$

Bayesian premium

$$B_{n+1} = \frac{a + \sum_{t=1}^n X_t}{a+b+n}$$

Credibility factor

$$z = \frac{n}{n+a+b}$$

## A.2 Binomial/beta case

$X|\Theta = \theta \sim \text{Binomial}(\nu, \theta)$

$\Theta \sim \text{Beta}(a, b)$

$\Theta|X_1 = x_1, \dots, X_n = x_n \sim \text{Beta}(\tilde{a}, \tilde{b})$

$$\tilde{a} = a + \sum_{t=1}^n x_t$$

$$\tilde{b} = b + n\nu - \sum_{t=1}^n x_t$$

Risk premium

$$\mu(\theta) = \nu\theta$$

Collective premium

$$m = \frac{\nu a}{a+b}$$

Bayesian premium

$$B_{n+1} = \frac{\nu(a + \sum_{t=1}^n X_t)}{a+b+n\nu}$$

Credibility factor

$$z = \frac{n}{n + (a+b)/\nu}$$

## A.3 Geometric/Beta case

$X|\Theta = \theta \sim \text{Geometric}(\theta)$

$\Theta \sim \text{Beta}(a, b)$

$\Theta|X_1 = x_1, \dots, X_n = x_n \sim \text{Beta}(\tilde{a}, \tilde{b})$

$$\tilde{a} = a + n$$

$$\tilde{b} = b + \sum_{t=1}^n x_t$$

Risk premium

$$\mu(\theta) = \frac{1-\theta}{\theta}$$

Collective premium

$$m = \frac{b}{a-1}$$

Bayesian premium

$$B_{n+1} = \frac{b + \sum_{t=1}^n X_t}{a + n - 1}$$

Credibility factor

$$z = \frac{n}{n + a - 1}$$

#### A.4 Negative binomial/Beta case

$X|\Theta = \theta \sim \text{Negative binomial}(r, \theta)$

$\Theta \sim \text{Beta}(a, b)$

$\Theta|X_1 = x_1, \dots, X_n = x_n \sim \text{Beta}(\tilde{a}, \tilde{b})$

$$\tilde{a} = a + nr$$

$$\tilde{b} = b + \sum_{t=1}^n x_t$$

Risk premium

$$\mu(\theta) = \frac{r(1-\theta)}{\theta}$$

Collective premium

$$m = \frac{rb}{a-1}$$

Bayesian premium

$$B_{n+1} = \frac{r(b + \sum_{t=1}^n X_t)}{a + nr - 1}$$

Credibility factor

$$z = \frac{n}{n + (a-1)/r}$$

#### A.5 Poisson/Gamma case

$X|\Theta = \theta \sim \text{Poisson}(\theta)$

$\Theta \sim \text{Gamma}(\alpha, \lambda)$

$\Theta|X_1 = x_1, \dots, X_n = x_n \sim \text{Gamma}(\tilde{\alpha}, \tilde{\lambda})$

$$\tilde{\alpha} = \alpha + \sum_{t=1}^n x_t$$

$$\tilde{\lambda} = \lambda + n$$

Risk premium

$$\mu(\theta) = \theta$$



Collective premium

$$m = \frac{\alpha}{\lambda}$$

Bayesian premium

$$B_{n+1} = \frac{\alpha + \sum_{t=1}^n X_t}{\lambda + n}$$

Credibility factor

$$z = \frac{n}{n + \lambda}$$

## A.6 Exponential/Gamma case

$X|\Theta = \theta \sim \text{Exponential}(\theta)$

$\Theta \sim \text{Gamma}(\alpha, \lambda)$

$\Theta|X_1 = x_1, \dots, X_n = x_n \sim \text{Gamma}(\tilde{\alpha}, \tilde{\lambda})$

$$\tilde{\alpha} = \alpha + n$$

$$\tilde{\lambda} = \lambda + \sum_{t=1}^n x_t$$

Risk premium

$$\mu(\theta) = \frac{1}{\theta}$$

Collective premium

$$m = \frac{\lambda}{\alpha - 1}$$

Bayesian premium

$$B_{n+1} = \frac{\lambda + \sum_{t=1}^n X_t}{\alpha + n - 1}$$

Credibility factor

$$z = \frac{n}{n + \alpha - 1}$$

## A.7 Gamma/Gamma case

$X|\Theta = \theta \sim \text{Gamma}(\tau, \theta)$

$\Theta \sim \text{Gamma}(\alpha, \lambda)$

$\Theta|X_1 = x_1, \dots, X_n = x_n \sim \text{Gamma}(\tilde{\alpha}, \tilde{\lambda})$

$$\tilde{\alpha} = \alpha + n\tau$$

$$\tilde{\lambda} = \lambda + \sum_{t=1}^n x_t$$

Risk premium

$$\mu(\theta) = \frac{\tau}{\theta}$$

Collective premium

$$m = \frac{\tau\lambda}{\alpha - 1}$$

Bayesian premium

$$B_{n+1} = \frac{\tau(\lambda + \sum_{t=1}^n X_t)}{\alpha + n\tau - 1}$$

Credibility factor

$$z = \frac{n}{n + (\alpha - 1)/\tau}$$

## A.8 Normal/Normal case

$X|\Theta = \theta \sim \text{Normal}(\theta, \sigma_2^2)$

$\Theta \sim \text{Normal}(\mu, \sigma_1^2)$

$\Theta|X_1 = x_1, \dots, X_n = x_n \sim \text{Normal}(\tilde{\mu}, \tilde{\sigma}_1^2)$

$$\tilde{\mu} = \frac{\sigma_1^2 \sum_{t=1}^n x_t + \sigma_2^2 \mu}{n\sigma_1^2 + \sigma_2^2}$$

$$\tilde{\sigma}_1^2 = \frac{\sigma_1^2 \sigma_2^2}{n\sigma_1^2 + \sigma_2^2}$$

Risk premium

$$\mu(\theta) = \theta$$

Collective premium

$$m = \mu$$

Bayesian premium

$$B_{n+1} = \frac{\sigma_1^2 \sum_{t=1}^n X_t + \sigma_2^2 \mu}{n\sigma_1^2 + \sigma_2^2}$$

Credibility factor

$$z = \frac{n}{n + \sigma_2^2/\sigma_1^2}$$

## References

- H. Belhadj, V. Goulet, and T. Ouellet. On parameter estimation in hierarchical credibility. *ASTIN Bulletin*, 39(2), 2009.
- H. Bühlmann. Experience rating and credibility. *ASTIN Bulletin*, 5:157–165, 1969.
- H. Bühlmann and A. Gisler. Credibility in the regression case revisited. *ASTIN Bulletin*, 27:83–98, 1997.

- H. Bühlmann and A. Gisler. *A course in credibility theory and its applications*. Springer, 2005. ISBN 3-5402575-3-5.
- H. Bühlmann and W. S. Jewell. Hierarchical credibility revisited. *Bulletin of the Swiss Association of Actuaries*, 87:35–54, 1987.
- H. Bühlmann and E. Straub. Glaubwürdigkeit für Schadensätze. *Bulletin of the Swiss Association of Actuaries*, 70:111–133, 1970.
- M. J. Goovaerts and W. J. Hoogstad. *Credibility theory*. Number 4 in Surveys of actuarial studies. Nationale-Nederlanden N.V., Netherlands, 1987.
- V. Goulet. Principles and application of credibility theory. *Journal of Actuarial Practice*, 6:5–62, 1998.
- C. A. Hachemeister. Credibility for regression models with application to trend. In *Credibility, theory and applications*, Proceedings of the Berkeley Actuarial Research Conference on Credibility, pages 129–163, New York, 1975. Academic Press.
- W. S. Jewell. The use of collateral data in credibility theory: a hierarchical model. *Giornale dell’Istituto Italiano degli Attuari*, 38:1–16, 1975.
- S. A. Klugman, H. H. Panjer, and G. Willmot. *Loss Models: From Data to Decisions*. Wiley, New York, 4 edition, 2012. ISBN 978-1-118-31532-3.
- E. Ohlsson. Simplified estimation of structure parameters in hierarchical credibility. Presented at the Zurich ASTIN Colloquium, 2005. URL <http://www.actuaries.org/ASTIN/Colloquia/Zurich/Ohlsson.pdf>.