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A Note on The Evolution of Preferences

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A Note on *The Evolution of Preferences*

Oliver Enrique Pardo Reinoso

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Abstract

This note checks the robustness of a surprising result in Dekel et al. (2007). The result states that strict Nash equilibria might cease to be evolutionary stable when agents are able to observe the opponent's preferences with a very low probability. This note shows that the result is driven by the assumption that there is no risk for the observed preferences to be mistaken. In particular, when a player may observe a signal correlated with the opponent's preferences, but the signal is noisy enough, all strict Nash equilibria are evolutionary stable.

1 Introduction

This note expands the evolutionary game theory model of Dekel et al. (2007) in order to check the robustness of their results. In their model, each player has a probability $p \in [0, 1]$ of observing the preferences of the opponent they are matched with. With complementary probability, they observe nothing. Their model yields the following results. Firstly, any efficient strict Nash equilibrium is evolutionary stable. Secondly, when p is close enough to 1, a pure strategy profile has to be efficient in order to be stable.[1] Thirdly, when $p = 0$, any strict Nash equilibrium is stable. This last result, however, is not continuous: there are strict Nash equilibria which are not stable for very low levels of p.[2] This note checks the robustness of these results when there is a probability that the observed preferences do not correspond to the opponents' actual preferences. It shows that the lack of stability of strict Nash equilibria for very low levels of p is driven by the assumption that the signals that players receive on the opponents' preferences are fully accurate. When the signals are noisy enough, all strict equilibria are stable.

The setup of the extended model is as follows. A large but finite number of players are randomly and uniformly matched with each other. In each match, each player observes a signal from her opponent. With a probability p, the player observes the opponent's preferences. With probability q, the player observes some preferences randomly drawn from the

[1] This result is reminiscent of the "secret handshake" result of Robson (1990) and the "information leaks lead to cooperation" result of Matsui (1989).

[2] In the context of strict coordination games, these results lead the authors to side in favor of the selection of payoff-dominant equilibria (Harsanyi and Selten (1988)) as opposed to risk-dominant equilibria (Carlsson and Van Damme (1993), Ellison (1993), Kandori et al. (1993)).

population. With probability $1-p-q$, the player observes no signal. Players in a match play a Bayesian Nash equilibrium (BNE) of an incomplete information game, where the type of each player is given by her preferences and the signal she observes. The outcome of the match induces some payoff in terms of fitness. However, this fitness function does not necessarily represent players' preferences. Generally speaking, an outcome is stable if no entrants -players with preferences different from those of incumbents- induce a BNE far away from the BNE previously played among incumbents and no entrants outperform incumbents in terms of the average fitness obtained across matches.

The results of the extended model are as follows. Just as in Dekel et al. (2007), efficient strict Nash equilibria are always stable (Proposition 1). Additionally, efficiency is a necessary condition for stability when the frequency $(p+q)$ and the precision (p/q) of the signal are high enough (Proposition 2). Furthermore, when the precision of the signal is low enough, (a) a pure-strategy profile has to be a Nash equilibrium to be stable, and (b) any strict Nash equilibrium is stable (Proposition 3). This last result implies that, as long as signals are noisy enough, the stability of strict Nash equilibria is robust to the introduction of low frequency signals.

The rest of this note is organized as follows. Section 2 presents the setup of the extended model. Section 3 defines the stability concept. Section 4 presents the results regarding the necessary and sufficient conditions for the stability of pure-strategy profiles. Section 5 presents some final comments. The proofs of all propositions are presented in the appendix.

2 Model

The underlying structure of the model is given by a symmetric game $G=(A,\pi)$ where A is a finite action set, $\triangle$ is the simplex on A and $\pi \in \mathbb{R}^{\triangle^2}$ is an **objective fitness** function. In particular, $\pi(\alpha,\beta)$ is the fitness players get when their strategy is $\alpha \in \triangle$ and their opponent's is $\beta \in \triangle$. Throughout the note, Latin letters are used instead of Greek letters when referring to pure strategies.

2.1 Preferences

Preferences over outcomes are not necessarily represented by a positive affine transformation of π. Still, preferences will be subject to evolutionary pressures that will define their chances of survival. The question is which kind of preferences will prevail.[3] The preferences are represented by von Neumann-Morgenstern (vNM) **utility** functions. Let $U \subset \mathbb{R}^{\triangle^2}$ be the set of vNM functions. For players with a utility function $u \in U$, the utility they get when their strategy is α and their opponent's is β is $u(\alpha,\beta)$. Since u does not have to be a positive affine transformation of π, preferences do not necessarily rank outcomes the same way as fitness does. The fitness function π will reappear in the analysis when assessing which preferences are able to survive evolutionary pressures.

[3] This is known as the *indirect approach* of evolutionary game theory, pioneered by Güth and Yaari (1992) and Güth (1995). In this approach, individuals are endowed with preferences, rather than with behavioral strategies that they blindly follow. Natural selection operates on preferences, not on strategies. The strategic outcome induced by a distribution of preferences is be part of the definition of stability.

2.2 Signals

The environment is populated by a large but finite number of players. Players are randomly matched with each other an infinite number of times. In each match, players play a game whose action set is given by A. For every match, players may receive a signal that helps them to update their beliefs regarding their opponent's preferences. This signal is private information. Consider a given distribution μ over utility functions in the existing population. Denote by C_μ the support of μ. The signal a player receives comes from the set:

$$X_\mu := C_\mu \cup \emptyset$$

The following events may arise when a player is matched with another: (a) with probability p, the player observes her opponent's preferences, (b) with probability q, the player observes preferences that are drawn according to μ, and (c) with probability $1 - p - q$, the player receives no signal. Throughout the note, it is assumed that p and q are both positive. This means that, unless the population is monomorphic, players can never be sure about the preferences of their opponents.[4] The model of Dekel et al. (2007) corresponds to the limiting case where $q = 0$.

2.3 Beliefs

In order to choose the best strategy according to their preferences, players have to form expectations about the signal their opponent receives. Let $g_\mu(x|u)$ be the probability that the opponent receives a signal $x \in X_\mu$ when the player's utility function is $u \in C_\mu$. We have:

$$g_\mu(x|u) = \begin{cases} p + q\mu\left(x\right) & \text{if } x = u \\ q\mu\left(x\right) & \text{if } x \in C_\mu/\{u\} \\ 1 - p - q & \text{if } x = \emptyset \end{cases}$$

Simultaneously, players try to infer their opponent's preferences from the signal they receive. It is assumed that players know the distribution of preferences in the population and that this is common knowledge. Let $f_\mu(u|x)$ be the probability that an opponent has preferences represented by $u \in C_\mu$, conditional on a signal $x \in X_\mu$. Bayes rule implies

$$f_\mu(u|x) = \begin{cases} \mu\left(u\right) & \text{for } x = \emptyset \\ \frac{p + q\mu(u)}{p+q} & \text{for } x \in C_\mu \end{cases}$$

2.4 Equilibrium play

A players private information in each match is given by $(u, y) \in C_\mu \times X_\mu$, where u represents their preferences and y the signal they receive from their opponent. Let $b : C_\mu \times X_\mu \to \triangle$ be a behavioral function such that $b(u, y)$ denotes the strategy played by agents with utility $u \in C_\mu$ when they receive a signal $y \in X_\mu$.

[4] A population is monomorphic if C_μ is a singleton.

It is assumed that natural selection operates long after individuals have learned the preferences supported by μ, their frequency and the strategies played by each type in $C_\mu \times X_\mu$. Rationality implies that players will play a Bayesian Nash Equilibrium (BNE)[5]:

Definition 1 *A function b is a Bayesian Nash Equilibrium for a given a distribution over preferences μ if*

$$b(u,y) \in \arg\max_{\alpha \in \triangle} \sum_{v \in C_\mu} \sum_{x \in X_\mu} u\left(\alpha, b(v,x)\right) f_\mu(v|y) g_\mu(x|u)$$

for $(u,y) \in C_\mu \times X_\mu$.

The set of BNE for a distribution of preferences μ is denoted by $\mathbb{B}(\mu)$. The tuple (μ, b) constitutes a *configuration* if $b \in \mathbb{B}(\mu)$. A configuration describes, not only the distribution of players' preferences, but also the strategies players use. The distribution over strategies induced by a configuration is known as its *outcome*.

3 Stability

The evolutionary feasibility of a distribution of preferences depends on the performance of the incumbent preferences relative to the mutant preferences that may enter the population. The performance of preferences is measured in terms of fitness. In particular, it is measured by the *average fitness* obtained after infinitely many rounds of matching. Denote by $\Pi_{(\mu,b)}(u)$ the average fitness players with preferences $u \in C_\mu$ get under a configuration (μ, b). When receiving a signal $y \in X_\mu$, a player with utility function u plays $b(u,y)$. The expected fitness from this play is

$$\sum_{v \in C_\mu} \sum_{x \in X_\mu} \pi\left(b(u,y), b(v,x)\right) g_\mu\left(x|u\right) \mu(v)$$

The average fitness for players with utility function u is obtained by averaging their expected fitness across the signals they receive:

$$\Pi_{(\mu,b)}(u) = \sum_{v \in C_\mu} \sum_{y \in X_\mu} \sum_{x \in X_\mu} \pi\left(b(u,y), b(v,x)\right) g_\mu\left(x|u\right) g_\mu\left(y|v\right) \mu(v)$$

The appearance of mutant preferences can disturb the equilibrium being played by incumbents. It is assumed that the equilibrium played after mutants' entrance is such that incumbents keep playing the same action when observing a signal corresponding to incumbents. Formally, consider a distribution $\tilde{\mu}$ whose support may or may not include preferences in the support of μ. An equilibrium $\tilde{b}$ is *focal* relative to the configuration (μ, b) if incumbents (those with utility function in C_μ) keep playing the same strategy when facing any signal in X_μ. The subset of $\mathbb{B}(\tilde{\mu})$ which is focal relative to (μ, b) is denoted by

$$\mathbb{F}_{(\mu,b)}\left(\tilde{\mu}\right) := \left\{ \tilde{b} \in \mathbb{B}\left(\tilde{\mu}\right) : \ \tilde{b}(u,y) = b(u,y) \text{ for all } (u,y) \in C_\mu \times X_\mu \right\}$$

[5]For an environment with no observability where people play self-confirming equilibria, see Gamba (2011)

In order to define stability, some additional notation has to be previously introduced. Consider a configuration (μ, b). Assume that a fraction $\epsilon \in (0,1)$ of the population mutates and switches to preferences represented by a vNM function $\tilde{u}$. After mutation, the distribution of preferences is given by

$$\tilde{\mu}(u) = \begin{cases} \epsilon & \text{if } u = \tilde{u} \\ (1-\epsilon)\mu(u) & \text{otherwise} \end{cases} \tag{1}$$

for $u \in U$. For a given $\tilde{u}$ and a given pre-entry distribution μ, the set $\mathcal{M}_{\bar{\epsilon}}(\mu, \tilde{u})$ denotes the set of post-entry distributions in which the fraction of entrants is no larger than $\bar{\epsilon} \in (0,1)$. In other words, the distribution $\tilde{\mu}$ is in $\mathcal{M}_{\bar{\epsilon}}(\mu, \tilde{u})$ if there is $\epsilon \in [0, \bar{\epsilon}]$ such that equation (1) holds.

A formal definition can be given now with the help of the notation introduced so far:

Definition 2 *A configuration (μ, b) is stable if there is $\bar{\epsilon} \in (0,1)$ such that, for every $\tilde{u} \in U$,*

$$\tilde{\mu} \in \mathcal{M}_{\bar{\epsilon}}(\mu, \tilde{u})$$

implies

1. $\mathbb{F}_{(\mu,b)}(\tilde{\mu}) \neq \emptyset$
2. $\Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) \leq \Pi_{(\tilde{\mu},\tilde{b})}(u)$ *for* $\tilde{b} \in \mathbb{F}_{(\mu,b)}(\tilde{\mu})$ *and* $u \in C_\mu$

A strategy in $\triangle$ is said to be stable if it is the outcome of a stable configuration. A distribution of preferences μ is stable if there is $b \in \mathbb{B}(\mu)$ such that the configuration (μ, b) is stable.

Generally speaking, stability means that, at least when the fraction of mutants is small, the following conditions hold: 1) there is at least one post-entry equilibrium in which mutants do not disturb the behavior of incumbents and 2) entrants do not outperform incumbents in any of these equilibria.[6]

4 Results

Define $p+q$ as the *frequency* of the signal and define p/q as the *precision* of the signal. In general, when the frequency and the precision of signals are high, stability favors the efficient outcomes of G. When the precision is low, stability favors the Nash equilibrium outcomes of G.

Definition 3 *A strategy profile $(\alpha, \alpha) \in \triangle^2$ is an efficient outcome of G if*

$$\pi(\alpha, \alpha) \geq \pi(\beta, \beta)$$

for all $(\beta, \beta) \in \triangle^2$.

[6]The stability concept in definition 2 is slightly more demanding than the one provided by Dekel et al. (2007). In Definition 2, if the set of focal equilibria is empty, the configuration is not stable. In Dekel et al. (2007), if the set of focal equilibria is empty, stability demands that entrants do not outperform incumbents in any BNE "close" to b. The results of this note can be extended to the latter and more relaxed definition of stability.

4.1 Stability of efficient strict Nash equilibria

The first result shows the robustness of efficient strict Nash equilibria to any kind of noise: no matter what values p and q take, an outcome which is an efficient strict Nash equilibrium of G is a stable outcome:

Proposition 1 *If (a, a) is both a strict Nash equilibrium and an efficient outcome of G, then it is stable.*

Proof: See Appendix.

The stability of a pure-strategy profile (a, a) which is both efficient and a strict Nash equilibrium can be supported by a monomorphic population for whom playing a is a dominant strategy. Entrants who are not playing a for all signals they receive will not outperform incumbents, either because they will not be best-responding to incumbents, or because they will not be able to coordinate among themselves on an outcome more efficient than (a, a).

4.2 Stability under high frequency and high precision

The following result shows that efficiency is a necessary condition for stability when *both* the precision and the frequency of the signal are high enough. To show this, define $\mathcal{N}(\bar{\delta})$ as the set of probability vectors whose difference from the situation of full observability is no higher than $\bar{\delta} \in (0, 1)$:

$$\mathcal{N}(\bar{\delta}) := \left\{(q, p) \in (0, 1)^2 \ : \ p + q \leq 1 \text{ and } |(q, p) - (0, 1)| \leq \bar{\delta}\right\}$$

Proposition 2 *If (a, a) is not an efficient outcome of G, then there is a $\bar{\delta} \in (0, 1)$ such that a is not a stable outcome for any $(q, p) \in \mathcal{N}(\bar{\delta})$.*

Proof: See Appendix.

Intuitively, consider a configuration that induces a non-efficient play (a, a). When both the signal frequency and the signal precision are high enough, there are entrants who will usually play a against incumbents and will usually play the efficient strategy against entrants. The proof relies on showing that there is a focal equilibrium in which the incumbents keep playing a regardless of the signal. If this is the case, the entrants will achieve a higher average fitness than the incumbents.

4.3 Stability under low precision

The following proposition states that, when the signal is very noisy, (a) a pure-strategy profile has to be a Nash equilibrium to be stable, and (b) any strict Nash equilibrium is stable. This result is *independent* of the frequency of the signal. Formally, define $\sigma := p/q$ as the precision of the signal:

Proposition 3

(a) If (a, a) is not a Nash equilibrium of G, then there is a $\bar{\sigma} \in (0, 1)$ such that a is not stable for any $\sigma \in [0, \bar{\sigma})$.

(b) If (a, a) is a strict Nash equilibrium of G, then there is a $\bar{\sigma} \in (0, 1)$ such that a is stable for any $\sigma \in [0, \bar{\sigma})$.

PROOF: See Appendix.

The intuition for the necessity result in (a) is as follows: If (a, a) is not a Nash equilibrium, then there is a strategy $\tilde{a}$ that offers a higher fitness than a when the opponent plays a. Consider an entrant for whom playing $\tilde{a}$ is a dominant strategy. In any focal equilibria, incumbents play a when receiving a signal corresponding to an incumbent. Since the signal is almost uninformative, this implies that entrants will rarely get a payoff different than $\pi(\tilde{a}, a)$. Meanwhile, since the fraction of entrants is small, incumbents will usually get $\pi(a, a)$. Since $\pi(\tilde{a}, a) > \pi(a, a)$, the entrants will take over the population.

In order to give an intuition for the sufficiency result in (b), consider a population for whom playing a is a dominant strategy. In order to have any chance of outperforming the incumbents, entrants will have to play a when receiving no signal and to play a when receiving the signal that their opponent is an incumbent. Suppose that an entrant receives a signal that her opponent is a fellow entrant. Since the signals is very noisy and the fraction of entrants is small, she is most likely to be facing an incumbent who plays a. Therefore, if she responds be playing something other than a, she will achieve a lower fitness than incumbents, since (a, a) is a strict Nash equilibrium of G.

In Dekel et al. (2007), the authors give an example of a strict coordination game in which there is a (risk-dominant) strict Nash equilibrium that is not stable even for negligibly but positive levels of observability. In contrast, propositions 1 and 3 imply:

Corollary 1 *For strict coordination games, (a) the payoff-dominant equilibrium is always stable, and (b) the risk-dominant equilibrium is stable if the signal precision is sufficiently low.*

In other words, the stability of a risk-dominant equilibrium that is not pay-off dominant is actually robust to the introduction of noisy signals. The significance of this result is that the equilibrium selection in favor of payoff-dominant equilibria is not as appealing as it seems when players can make inferences about opponents' preferences. Once it is acknowledged that the information regarding opponents' preferences could be very noisy, the selection against payoff-dominated equilibria ceases to be straightforward.

5 Conclusion

It is clearly unrealistic to assume that preferences can be perfectly observed. On the other hand, it is also unrealistic to assume that no information regarding the opponents' preferences exists. This note has studied the intermediate case where there is some imprecise information regarding opponents' preferences. No matter whether the signals on the opponent's preferences are noisy or not, there are always stable preferences that support efficient strict Nash equilibria. Whenever the signals are accurate and frequent enough, a pure strategy profile has to be efficient to be stable. Finally, the note has checked the robustness of the most puzzling result of Dekel et al. (2007), which is that strict Nash equilibria might cease to be stable when a low-frequency signal is introduced. Once information distortions are acknowledged, the "folk" result is reinstated: if the signal is noisy enough, all strict Nash equilibria remain stable.

An assumption made throughout this note is that the probability of observing no signal is independent of types. It seems plausible, however, that observing a signal becomes more likely as the divergence of preferences decreases. In such a setup, it is conjectured that there will be stable configurations where players: i) play an efficient strategy if the opponent is perceived as having similar preferences, and ii) play a Nash strategy in any other case.

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6 Appendix

Proposition 1 *If (a,a) is both a strict Nash equilibrium and an efficient outcome of G, then it is stable.*

Proof 1 *Consider a configuration (μ,b) where $C_\mu=\{u\}$ and playing a is a strictly dominant strategy for u. Denote by $\tilde{u}$ the utility function for an entrant. Consider a focal equilibrium $\tilde{b}\in\mathbb{F}_{(\mu,b)}(\tilde{\mu})$ such that $\tilde{\mu}$ is such that (1) holds for some $\epsilon\in(0,1)$. Since a is strictly dominant for u, $\mathbb{F}_{(\mu,b)}(\tilde{\mu})$ is not empty and $\tilde{b}(u,x)=a$ for $x\in\{u,\tilde{u},\emptyset\}$.*

To shorten notation, denote the strategies an entrant follows when receiving signals u, $\tilde{u}$ and $\emptyset$ by

$$\begin{aligned}\gamma_u &:= \tilde{b}(\tilde{u},u)\\ \gamma_{\tilde{u}} &:= \tilde{b}(\tilde{u},\tilde{u})\\ \gamma_\emptyset &:= \tilde{b}(\tilde{u},\emptyset)\end{aligned}$$

The fitness for incumbents will be

$$\Pi_{(\tilde{\mu},\tilde{b})}(u)=(1-\epsilon)\pi(a,a)+\epsilon\begin{pmatrix}p+q(1-\epsilon)\\ q\epsilon\\ 1-p-q\end{pmatrix}'\begin{pmatrix}\pi(a,\gamma_u)\\ \pi(a,\gamma_{\tilde{u}})\\ \pi(a,\gamma_\emptyset)\end{pmatrix} \tag{2}$$

whereas the fitness for the entrant will be

$$\begin{aligned}\Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u})=(1-\epsilon)&\begin{pmatrix}p+q(1-\epsilon)\\ q\epsilon\\ 1-p-q\end{pmatrix}'\begin{pmatrix}\pi(\gamma_u,a)\\ \pi(\gamma_{\tilde{u}},a)\\ \pi(\gamma_\emptyset,a)\end{pmatrix}\\ +\epsilon&\begin{pmatrix}p+q\epsilon\\ q\,(1-\epsilon)\\ 1-p-q\end{pmatrix}'\begin{pmatrix}\pi(\gamma_{\tilde{u}},\gamma_{\tilde{u}}) & \pi(\gamma_{\tilde{u}},\gamma_u) & \pi(\gamma_{\tilde{u}},\gamma_\emptyset)\\ \pi(\gamma_u,\gamma_{\tilde{u}}) & \pi(\gamma_u,\gamma_u) & \pi(\gamma_u,\gamma_\emptyset)\\ \pi(\gamma_\emptyset,\gamma_{\tilde{u}}) & \pi(\gamma_\emptyset,\gamma_u) & \pi(\gamma_\emptyset,\gamma_\emptyset)\end{pmatrix}\begin{pmatrix}p+q\epsilon\\ q\,(1-\epsilon)\\ 1-p-q\end{pmatrix}\end{aligned} \tag{3}$$

Consider the case where either γ_u or $\gamma_\emptyset$ are not equal to a. Subtracting (3) from (2) yields

$$\Pi_{(\tilde{\mu},\tilde{b})}(u)-\Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u})=\pi(a,a)-(p+q)\pi(\gamma_u,a)-(1-p-q)\pi(\gamma_\emptyset,a)+o(\epsilon) \tag{4}$$

Since (a,a) is a strict Nash equilibrium, there is ϵ small enough such that (4) is positive.

Consider the case where γ_u and $\gamma_\emptyset$ are both equal to a. Subtracting (3) from (2) yields

$$\begin{aligned}\Pi_{(\tilde{\mu},\tilde{b})}(u)-\Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) &= \epsilon q\left(\pi(a,a)-\pi(\gamma_{\tilde{u}},a)\right)\\ &+ \epsilon p^2\left(\pi(a,a)-\pi(\gamma_{\tilde{u}},\gamma_{\tilde{u}})\right)\\ &+ \epsilon p2\,(1-p)\left(\pi(a,a)-\frac{1}{2}\pi(a,\gamma_{\tilde{u}})-\frac{1}{2}\pi(\gamma_{\tilde{u}},a)\right)+o(\epsilon^2)\end{aligned} \tag{5}$$

The first term on the right hand side of (4) is positive because (a,a) is a strict Nash equilibrium. The second and third terms of (4) are positive because (a,a) is an efficient outcome. Hence, there is an ϵ small enough such that (5) is positive.

It has been shown that, for every entrant $\tilde{u} \in U$, there is ϵ small enough such that they do not outperform incumbents. Still, it has to be shown that there is a uniform barrier $\bar{\epsilon}$ for which the fitness of any entrant cannot be higher than the fitness of incumbents when the fraction of entrants is no higher than $\bar{\epsilon}$. To show this, rewrite the strategy for the entrants in the following way. For $x \in \{u, \tilde{u}, \emptyset\}$, find $t_x \in [0,1]$ and $\beta_x \in \triangle$ such that

$$\gamma_x = t_x a + (1 - t_x)\beta_x$$

where the support of β_x does not include a.

The fitness for the incumbents under configuration $(\tilde{\mu}, \tilde{b})$ is given by

$$\begin{aligned} \Pi_{(\tilde{\mu},\tilde{b})}(u) &= (1-\epsilon)\pi(a,a) \\ &+ \epsilon \begin{pmatrix} p+q(1-\epsilon) \\ q\epsilon \\ 1-p-q \end{pmatrix}' \begin{pmatrix} t_u \\ t_{\tilde{u}} \\ t_{\emptyset} \end{pmatrix} \pi(a,a) \\ &+ \epsilon \begin{pmatrix} p+q(1-\epsilon) \\ q\epsilon \\ 1-p-q \end{pmatrix}' \begin{pmatrix} 1-t_u & 0 & 0 \\ 0 & 1-t_{\tilde{u}} & 0 \\ 0 & 0 & 1-t_{\emptyset} \end{pmatrix} \begin{pmatrix} \pi(a,\beta_u) \\ \pi(a,\beta_{\tilde{u}}) \\ \pi(a,\beta_{\emptyset}) \end{pmatrix} \end{aligned}$$

whereas the fitness for the entrants is given by

$$\begin{aligned} \Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) &= (1-\epsilon) \begin{pmatrix} p+q(1-\epsilon) \\ q\epsilon \\ 1-p-q \end{pmatrix}' \begin{pmatrix} t_u \\ t_{\tilde{u}} \\ t_{\emptyset} \end{pmatrix} \pi(a,a) \\ &+ (1-\epsilon) \begin{pmatrix} p+q(1-\epsilon) \\ q\epsilon \\ 1-p-q \end{pmatrix}' \begin{pmatrix} 1-t_u & 0 & 0 \\ 0 & 1-t_{\tilde{u}} & 0 \\ 0 & 0 & 1-t_{\emptyset} \end{pmatrix} \begin{pmatrix} \pi(\beta_u,a) \\ \pi(\beta_{\tilde{u}},a) \\ \pi(\beta_{\emptyset},a) \end{pmatrix} \\ &+ \epsilon \begin{pmatrix} p+q\epsilon \\ q\,(1-\epsilon) \\ 1-p-q \end{pmatrix}' \begin{pmatrix} \pi(\gamma_{\tilde{u}},\gamma_{\tilde{u}}) & \pi(\gamma_{\tilde{u}},\gamma_u) & \pi(\gamma_{\tilde{u}},\gamma_{\emptyset}) \\ \pi(\gamma_u,\gamma_{\tilde{u}}) & \pi(\gamma_u,\gamma_u) & \pi(\gamma_u,\gamma_{\emptyset}) \\ \pi(\gamma_{\emptyset},\gamma_{\tilde{u}}) & \pi(\gamma_{\emptyset},\gamma_u) & \pi(\gamma_{\emptyset},\gamma_{\emptyset}) \end{pmatrix} \begin{pmatrix} p+q\epsilon \\ q\,(1-\epsilon) \\ 1-p-q \end{pmatrix} \end{aligned}$$

The difference between fitness payoffs is then given by:

$$\Pi_{(\tilde{\mu},\tilde{b})}(u) - \Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) =$$

$$\begin{pmatrix} p+q(1-\epsilon) \\ q\epsilon \\ 1-p-q \end{pmatrix}' \begin{pmatrix} 1-t_u & 0 & 0 \\ 0 & 1-t_{\tilde{u}} & 0 \\ 0 & 0 & 1-t_{\emptyset} \end{pmatrix} \begin{pmatrix} \pi(a,a)-\pi(\beta_u,a) \\ \pi(a,a)-\pi(\beta_{\tilde{u}},a) \\ \pi(a,a)-\pi(\beta_{\emptyset},a) \end{pmatrix}$$

$$-\epsilon \begin{pmatrix} p+q(1-\epsilon) \\ q\epsilon \\ 1-p-q \end{pmatrix}' \begin{pmatrix} 1-t_u & 0 & 0 \\ 0 & 1-t_{\tilde{u}} & 0 \\ 0 & 0 & 1-t_{\emptyset} \end{pmatrix} \begin{pmatrix} \pi(a,a)-\pi(\beta_u,a)-\pi(a,\beta_u) \\ \pi(a,a)-\pi(\beta_{\tilde{u}},a)-\pi(a,\beta_{\tilde{u}}) \\ \pi(a,a)-\pi(\beta_{\emptyset},a)-\pi(a,\beta_{\emptyset}) \end{pmatrix}$$

$$+\epsilon \begin{pmatrix} p+q(1-\epsilon) \\ q\epsilon \\ 1-p-q \end{pmatrix}' \begin{pmatrix} t_u \\ t_{\tilde{u}} \\ t_{\emptyset} \end{pmatrix} \pi(a,a)$$

$$-\epsilon \begin{pmatrix} p+q\epsilon \\ q\,(1-\epsilon) \\ 1-p-q \end{pmatrix}' \begin{pmatrix} \pi(\gamma_{\tilde{u}},\gamma_{\tilde{u}}) & \pi(\gamma_{\tilde{u}},\gamma_u) & \pi(\gamma_{\tilde{u}},\gamma_{\emptyset}) \\ \pi(\gamma_u,\gamma_{\tilde{u}}) & \pi(\gamma_u,\gamma_u) & \pi(\gamma_u,\gamma_{\emptyset}) \\ \pi(\gamma_{\emptyset},\gamma_{\tilde{u}}) & \pi(\gamma_{\emptyset},\gamma_u) & \pi(\gamma_{\emptyset},\gamma_{\emptyset}) \end{pmatrix} \begin{pmatrix} p+q\epsilon \\ q\,(1-\epsilon) \\ 1-p-q \end{pmatrix}$$

Efficiency implies that

$$\begin{pmatrix} p+q\epsilon \\ q\,(1-\epsilon) \\ 1-p-q \end{pmatrix}' \begin{pmatrix} \pi(\gamma_{\tilde{u}},\gamma_{\tilde{u}}) & \pi(\gamma_{\tilde{u}},\gamma_u) & \pi(\gamma_{\tilde{u}},\gamma_{\emptyset}) \\ \pi(\gamma_u,\gamma_{\tilde{u}}) & \pi(\gamma_u,\gamma_u) & \pi(\gamma_u,\gamma_{\emptyset}) \\ \pi(\gamma_{\emptyset},\gamma_{\tilde{u}}) & \pi(\gamma_{\emptyset},\gamma_u) & \pi(\gamma_{\emptyset},\gamma_{\emptyset}) \end{pmatrix} \begin{pmatrix} p+q\epsilon \\ q\,(1-\epsilon) \\ 1-p-q \end{pmatrix} \leq \pi(a,a)$$

Therefore

$$\Pi_{(\tilde{\mu},\tilde{b})}(u) - \Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) \geq$$

$$\begin{pmatrix} p+q(1-\epsilon) \\ q\epsilon \\ 1-p-q \end{pmatrix}' \begin{pmatrix} 1-t_u & 0 & 0 \\ 0 & 1-t_{\tilde{u}} & 0 \\ 0 & 0 & 1-t_{\emptyset} \end{pmatrix} \begin{pmatrix} \pi(a,a)-\pi(\beta_u,a) \\ \pi(a,a)-\pi(\beta_{\tilde{u}},a) \\ \pi(a,a)-\pi(\beta_{\emptyset},a) \end{pmatrix} -$$

$$\epsilon \begin{pmatrix} p+q(1-\epsilon) \\ q\epsilon \\ 1-p-q \end{pmatrix}' \begin{pmatrix} 1-t_u & 0 & 0 \\ 0 & 1-t_{\tilde{u}} & 0 \\ 0 & 0 & 1-t_{\emptyset} \end{pmatrix} \begin{pmatrix} 2\pi(a,a)-\pi(\beta_u,a)-\pi(a,\beta_u) \\ 2\pi(a,a)-\pi(\beta_{\tilde{u}},a)-\pi(a,\beta_{\tilde{u}}) \\ 2\pi(a,a)-\pi(\beta_{\emptyset},a)-\pi(a,\beta_{\emptyset}) \end{pmatrix}$$

Since (a,a) is a strict Nash equilibrium, there is an $\bar{\epsilon}$ such that the right hand side of this inequality is non-negative for all $\epsilon \in (0,\bar{\epsilon})$ and any $(t_x,\beta_x)_{x\in\{u,\tilde{u},\emptyset\}}$.

Proposition 2 *If (a,a) is not an efficient outcome of G, then there is a $\bar{\delta} \in (0,1)$ such that a is not a stable outcome for any $(q,p) \in \mathcal{N}(\bar{\delta})$.*

Proof 2 *Since a is not efficient, there is a $\beta \in \triangle$ such that there is a $\bar{\delta}$ such that*

$$\pi(a,a) < q\,(\pi(\beta,a)-\pi(a,a)) + \begin{pmatrix} p \\ 1-p \end{pmatrix}' \begin{pmatrix} \pi(\beta,\beta) & \pi(\beta,a) \\ \pi(a,\beta) & \pi(a,a) \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} \qquad (6)$$

for all $(q,p) \in \mathcal{N}(\bar{\delta})$.

The task is to find for every ϵ an entrant $\tilde{u}$ for which there is a focal equilibrium such that the entrant outperforms the incumbents. Consider a post-entry focal equilibrium in which the incumbents play a no matter the signal and entrants i) play a when perceiving an incumbent or receiving no signal and ii) play β when perceiving and entrant. Such a focal equilibrium exists when ϵ is small enough and when $\tilde{u}$ is chosen suitably.

Suppose a is a stable outcome for configuration (μ, b), where $b(u, x) = a$ for all $(u, x) \in C_\mu \times X_\mu$. The post-entry payoff for incumbents in this post-entry focal equilibrium is:

$$\Pi_{(\tilde{\mu},\tilde{b})}(u) = \pi(a,a) + \epsilon^2 q\left(\pi\left(a,\beta\right) - \pi\left(a,a\right)\right)$$

for $u \in C_\mu$. The only possibility for an incumbent to receive a payoff not equal to $\pi(a,a)$ is to encounter an entrant and being misperceived as an entrant, which is an event that occurs with probability proportional to ϵ^2.

Meanwhile, the payoff for entrants is given by:

$$\begin{aligned}\Pi_{(\tilde{\mu},\tilde{b})}\left(\tilde{u}\right) &= (1-\epsilon)\left((1-q\epsilon)\,\pi(a,a) + q\epsilon\pi(\beta,a)\right) \\ &+ \epsilon\begin{pmatrix} p+q\epsilon \\ 1-p-q\epsilon\end{pmatrix}^{'}\begin{pmatrix}\pi(\beta,\beta) & \pi(\beta,a) \\ \pi(a,\beta) & \pi(a,a)\end{pmatrix}\begin{pmatrix} p+q\epsilon \\ 1-p-q\epsilon\end{pmatrix},\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\Pi_{(\tilde{\mu},\tilde{b})}\left(\tilde{u}\right) &= (1-\epsilon)\pi(a,a) + \epsilon q\left(\pi(\beta,a) - \pi(a,a)\right) \\ &+ \epsilon\begin{pmatrix} p \\ 1-p\end{pmatrix}^{'}\begin{pmatrix}\pi(\beta,\beta) & \pi(\beta,a) \\ \pi(a,\beta) & \pi(a,a)\end{pmatrix}\begin{pmatrix} p \\ 1-p\end{pmatrix} + o(\epsilon^2).\end{aligned}$$

Stability implies:

$$\Pi_{(\tilde{\mu},\tilde{b})}\left(u\right) \geq \Pi_{(\tilde{\mu},\tilde{b})}\left(\tilde{u}\right)$$

Therefore:

$$\begin{aligned}\epsilon\pi(a,a) &\geq \epsilon q\left(\pi(\beta,a) - \pi(a,a)\right) \\ &+ \epsilon\begin{pmatrix} p \\ 1-p\end{pmatrix}^{'}\begin{pmatrix}\pi(\beta,\beta) & \pi(\beta,a) \\ \pi(a,\beta) & \pi(a,a)\end{pmatrix}\begin{pmatrix} p \\ 1-p\end{pmatrix} + o(\epsilon^2)\end{aligned}$$

This contradicts inequality (6) *when* $(q,p) \in \mathcal{N}(\bar{\delta})$

PROPOSITION 3

(a) If (a,a) is not a Nash equilibrium of G, then there is a $\bar{\sigma} \in (0,1)$ such that a is not stable for any $\sigma \in [0, \bar{\sigma})$.

(b) If (a,a) is a strict Nash equilibrium of G, then there is a $\bar{\sigma} \in (0,1)$ such that a is stable for any $\sigma \in [0, \bar{\sigma})$.

Proof 3 *(a) Since (a,a) is not a Nash equilibrium, there is a degenerate distribution $\tilde{a} \in \triangle$ such that*

$$\pi\left(a,a\right) < \pi\left(\tilde{a},a\right)$$

Therefore, there is $\bar{\sigma} \in (0,1)$ *such that, for any* $\sigma \in [0, \bar{\sigma})$,

$$\pi(a,a) - \left(\sigma q \pi\left(\tilde{a}, \underline{\beta}\right) + (1-\sigma q)\,\pi(\tilde{a}, a)\right) < 0 \tag{7}$$

where

$$\underline{\beta} \in \arg\min_{\beta \in \triangle} \pi(\tilde{a}, \beta)$$

Suppose there is a stable configuration (μ, b) *with outcome a. Consider an entrant* $\tilde{u}$ *for whom playing* $\tilde{a}$ *is a strictly dominant strategy. Since,* (μ, b) *is stable, there is a distribution* $\tilde{\mu}$ *as in (1) such that* $\mathbb{F}_{(\mu,b)}(\tilde{\mu})$ *is not empty for* ϵ *small enough.*

For $\tilde{b} \in \mathbb{F}_{(\mu,b)}(\tilde{\mu})$, *the average payoff for incumbents is given by*

$$\begin{aligned}
\sum_{u \in C_\mu} \Pi_{(\tilde{\mu},\tilde{b})}(u)\mu(u) &= (1-\epsilon)\begin{pmatrix} 1-q\epsilon \\ q\epsilon \end{pmatrix}' \begin{pmatrix} \pi(a,a) & X \\ Y & Z \end{pmatrix} \begin{pmatrix} 1-q\epsilon \\ q\epsilon \end{pmatrix} \pi(a,a) \\
&+ \epsilon(\sigma+\epsilon)\, q \sum_{v \in C_\mu} \pi\left(\tilde{b}(v,\tilde{u}), \tilde{a}\right)\mu(v) \\
&+ \epsilon\left(1-(\sigma+\epsilon)\,q\right)\pi(a,\tilde{a})
\end{aligned}$$

where

$$\begin{aligned}
X &:= \sum_{v \in C_\mu} \pi\left(\tilde{b}(v,\tilde{u}), a\right)\mu(v) \\
Y &:= \sum_{w \in C_\mu} \pi\left(a, \tilde{b}(w,\tilde{u})\right)\mu(w) \\
Z &:= \sum_{v \in C_\mu}\sum_{w \in C_\mu} \pi\left(\tilde{b}(v,\tilde{u}), \tilde{b}(w,\tilde{u})\right)\mu(v)\mu(w)
\end{aligned}$$

Hence

$$\sum_{u \in C_\mu} \Pi_{(\tilde{\mu},\tilde{b})}(u)\mu(u) = \pi(a,a) + o(\epsilon)$$

Meanwhile, the payoff for an entrant is given by

$$\begin{aligned}
\Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) &= (1-\epsilon)(\sigma+\epsilon)q \sum_{u \in C_\mu} \pi\left(\tilde{a}, \tilde{b}(u,\tilde{u})\right)\mu(u) \\
&+ (1-\epsilon)\left(1-(\sigma+\epsilon)\,q\right)\pi(\tilde{a}, a) \\
&+ \epsilon\pi(\tilde{a},\tilde{a})
\end{aligned}$$

Therefore:

$$\Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) \geq \left(\sigma q \pi\left(\tilde{a}, \underline{\beta}\right) + (1-\sigma q)\,\pi(\tilde{a}, a)\right) + o(\epsilon)$$

Stability implies that, for all ϵ *below certain threshold:*

$$\sum_{u \in C_\mu} \Pi_{(\tilde{\mu},\tilde{b})}(u)\mu(u) - \Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) \geq 0$$

This implies

$$\pi(a,a) - \left(\sigma q \pi\left(\tilde{a}, \underline{\beta}\right) + (1-\sigma q)\,\pi\left(\tilde{a}, a\right)\right) + o(\epsilon) \geq 0$$

For $\sigma \in [0, \bar{\sigma})$, this contradicts inequality (7).

(b) Consider a configuration (μ, b) where $C_\mu = \{u\}$ and playing a is a strictly dominant strategy for u. Denote by $\tilde{u}$ the utility function for an entrant. Consider a focal equilibrium $\tilde{b} \in \mathbb{F}_{(\mu,b)}(\tilde{\mu})$ such that $\tilde{\mu}$ is such that (1) holds for some $\epsilon \in (0,1)$. Since a is strictly dominant for u, $\mathbb{F}_{(\mu,b)}(\tilde{\mu})$ is not empty and $\tilde{b}(u,x) = a$ for $x \in \{u, \tilde{u}, \emptyset\}$.

To shorten the notation, denote the strategies an entrant follows when receiving signals u, $\tilde{u}$ and $\emptyset$ by

$$\gamma_u := \tilde{b}(\tilde{u}, u)$$
$$\gamma_{\tilde{u}} := \tilde{b}(\tilde{u}, \tilde{u})$$
$$\gamma_\emptyset := \tilde{b}(\tilde{u}, \emptyset)$$

Consider the case where γ_u or $\gamma_\emptyset$ are not equal to a. Since (a,a) is a strict Nash equilibrium of G, it follows that

$$(p+q)\,\pi\left(\gamma_u, a\right) + (1-(p+q))\,\pi\left(\gamma_\emptyset, a\right) < \pi\left(a, a\right)$$

If this is the case, an entrant who is not best responding with a when receiving signals u or $\emptyset$ will not be able to outperform the incumbents whenever the fraction of entrants is sufficiently small.

Consider the case where γ_u and $\gamma_\emptyset$ are equal to a. Decompose the entrant's strategy when receiving signal $\tilde{u}$ in the following way: find $t_{\tilde{u}} \in [0,1)$ and $\beta_{\tilde{u}} \in \triangle$ such that:

$$\gamma_{\tilde{u}} = t_{\tilde{u}} a + (1-t_{\tilde{u}})\beta_{\tilde{u}}$$

where the support of $\beta_{\tilde{u}}$ does not include a.

The fitness for the incumbent in the focal equilibrium is given by

$$\Pi_{(\tilde{\mu},\tilde{b})}(u) = (1-\epsilon)\pi\left(a,a\right) + \epsilon\left(\pi\left(a,a\right) + (1-t_{\tilde{u}})\,q\epsilon\left(\pi\left(a, \beta_{\tilde{u}}\right) - \pi\left(a,a\right)\right)\right)$$

meaning that not receiving a payoff $\pi(a,a)$ is an event of order ϵ^2: it requires being matched with an entrant and being misperceived as an entrant.

The fitness for the entrant is given by:

$$\begin{aligned}\Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) &= (1-\epsilon)\left(\pi\left(a,a\right) + (1-t_{\tilde{u}})\,q\epsilon\left(\pi\left(\beta_{\tilde{u}}, a\right) - \pi\left(a,a\right)\right)\right) \\ &+ \epsilon \begin{pmatrix} p+q\epsilon \\ 1-p-q\epsilon \end{pmatrix}' \begin{pmatrix} \pi(\gamma_{\tilde{u}}, \gamma_{\tilde{u}}) & \pi(a, \gamma_{\tilde{u}}) \\ \pi(\gamma_{\tilde{u}}, a) & \pi(a,a) \end{pmatrix} \begin{pmatrix} p+q\epsilon \\ 1-p-q\epsilon \end{pmatrix}\end{aligned}$$

Using the fact that $p = \sigma q$, the fitness for the entrant can be rewritten as:

$$\begin{aligned}\Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) &= (1-\epsilon)\pi\left(a,a\right) + (1-t_{\tilde{u}})\,q\epsilon\left(\pi\left(\beta_{\tilde{u}}, a\right) - \pi\left(a,a\right)\right) \\ &+ \epsilon \begin{pmatrix} (\sigma+\epsilon)q \\ 1-(\sigma+\epsilon)q \end{pmatrix}' \begin{pmatrix} \pi(\gamma_{\tilde{u}}, \gamma_{\tilde{u}}) & \pi(a, \gamma_{\tilde{u}}) \\ \pi(\gamma_{\tilde{u}}, a) & \pi(a,a) \end{pmatrix} \begin{pmatrix} (\sigma+\epsilon)q \\ 1-(\sigma+\epsilon)q \end{pmatrix}\end{aligned}$$

Therefore, the difference between payoffs will be given by:

$$\Pi_{(\tilde{\mu},\tilde{b})}(u) - \Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) = \epsilon\,(1-t_{\tilde{u}})\,q\,(\pi\,(a,a) - \pi\,(\beta_{\tilde{u}},a)) + \epsilon\left(\pi\,(a,a) - \begin{pmatrix} (\sigma+\epsilon)q \\ 1-(\sigma+\epsilon)q \end{pmatrix}' \begin{pmatrix} \pi(\gamma_{\tilde{u}},\gamma_{\tilde{u}}) & \pi(a,\gamma_{\tilde{u}}) \\ \pi(\gamma_{\tilde{u}},a) & \pi(a,a) \end{pmatrix} \begin{pmatrix} (\sigma+\epsilon)q \\ 1-(\sigma+\epsilon)q \end{pmatrix}\right)$$

The expression can be rewritten as:

$$\begin{aligned}
\Pi_{(\tilde{\mu},\tilde{b})}(u) - \Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) &= (1-t_{\tilde{u}})\,\epsilon q\,(\pi\,(a,a) - \pi\,(\beta_{\tilde{u}},a)) + \\
&\epsilon(\sigma+\epsilon)q(\sigma+\epsilon)q\left[\pi\,(a,a) - \begin{pmatrix} t_{\tilde{u}} \\ 1-t_{\tilde{u}} \end{pmatrix}' \begin{pmatrix} \pi\,(a,a) & \pi(a,\beta_{\tilde{u}}) \\ \pi(\beta_{\tilde{u}},a) & \pi(\beta_{\tilde{u}},\beta_{\tilde{u}}) \end{pmatrix} \begin{pmatrix} t_{\tilde{u}} \\ 1-t_{\tilde{u}} \end{pmatrix}\right] + \\
&\epsilon(\sigma+\epsilon)q(1-(\sigma+\epsilon)q)\left[\pi\,(a,a) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}' \begin{pmatrix} \pi\,(a,a) & \pi(a,\beta_{\tilde{u}}) \\ \pi(\beta_{\tilde{u}},a) & \pi(\beta_{\tilde{u}},\beta_{\tilde{u}}) \end{pmatrix} \begin{pmatrix} t_{\tilde{u}} \\ 1-t_{\tilde{u}} \end{pmatrix}\right] + \\
&\epsilon(\sigma+\epsilon)q(1-(\sigma+\epsilon)q)\left[\pi\,(a,a) - \begin{pmatrix} t_{\tilde{u}} \\ 1-t_{\tilde{u}} \end{pmatrix}' \begin{pmatrix} \pi\,(a,a) & \pi(a,\beta_{\tilde{u}}) \\ \pi(\beta_{\tilde{u}},a) & \pi(\beta_{\tilde{u}},\beta_{\tilde{u}}) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \qquad (8)
\end{aligned}$$

Since (a,a) is a strict Nash equilibrium, it follows that:

$$\pi\,(a,a) - \begin{pmatrix} t_{\tilde{u}} \\ 1-t_{\tilde{u}} \end{pmatrix}' \begin{pmatrix} \pi\,(a,a) & \pi(a,\beta_{\tilde{u}}) \\ \pi(\beta_{\tilde{u}},a) & \pi(\beta_{\tilde{u}},\beta_{\tilde{u}}) \end{pmatrix} \begin{pmatrix} t_{\tilde{u}} \\ 1-t_{\tilde{u}} \end{pmatrix} > \pi\,(a,a) - \begin{pmatrix} t_{\tilde{u}} \\ 1-t_{\tilde{u}} \end{pmatrix}' \begin{pmatrix} \pi\,(a,a) & \pi(\bar{\beta},\beta_{\tilde{u}}) \\ \pi\,(a,a) & \pi(\bar{\beta},\beta_{\tilde{u}}) \end{pmatrix} \begin{pmatrix} t_{\tilde{u}} \\ 1-t_{\tilde{u}} \end{pmatrix}$$

where

$$\bar{\beta} := \arg\max_{\zeta\in\{a,\beta_{\tilde{u}}\}} \pi(\zeta,\beta_{\tilde{u}})$$

Substituting this inequality into (8) yields:

$$\begin{aligned}
\Pi_{(\tilde{\mu},\tilde{b})}(u) - \Pi_{(\tilde{\mu},\tilde{b})}(\tilde{u}) \;>\; & (1-t_{\tilde{u}})\,\epsilon q\,(\pi\,(a,a) - \pi\,(\beta_{\tilde{u}},a)) \qquad (9) \\
+\; & (1-t_{\tilde{u}})\,\epsilon(\sigma+\epsilon)^2 q^2\left(\pi(a,a) - \pi(\bar{\beta},\beta_{\tilde{u}})\right) \\
+\; & (1-t_{\tilde{u}})\,\epsilon(\sigma+\epsilon)q(1-(\sigma+\epsilon)q)\,(\pi\,(a,a) - \pi(a,\beta_{\tilde{u}})) \\
+\; & (1-t_{\tilde{u}})\,\epsilon(\sigma+\epsilon)q(1-(\sigma+\epsilon)q)\,(\pi\,(a,a) - \pi(\beta_{\tilde{u}},a))
\end{aligned}$$

Therefore, we can find a suitable threshold for σ such that the right-hand side of (9) is non-negative for any $t_{\tilde{u}} \in [0,1]$ and ϵ small enough.